SECTION 5.3: THE FUNDAMENTAL THEOREM OF CALCULUS

Before we jump into the Fundamental Theorem of Calculus, we need to review some previous results.

RECALL:

- **DIFFERENTIABLITY IMPLIES CONTIUITY**: If $f'(a)$ exists, then f is continuous at $x = a$.
- **MEAN VALUE THEOREM**:

  If F is continuous over an interval $[a, b]$ and differentiable on (a, b) then there is a value c in (a, b) so that:

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

- **DEFINITION OF THE DEFINITE INTEGRAL**: If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

  where Δ runs through all partitions of $[a, b]$ and x_i^* lies in the interval $[x_{i-1}, x_i]$

THE FUNDAMENTAL THEOREM OF CALCULUS (EVALUATION OF DEFINITE INTEGRALS):
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

PROOF: (VIDEO) Let $\Delta = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$.

CLAIM: F satisfies the conditions of the Mean Value Theorem on each subinterval $[x_{i-1}, x_i]$.

Since F is an antiderivative of f on $[a, b]$ and f is continuous on $[a, b]$, $F'(x) = f(x)$ for all x in $[a, b]$, and, in particular, for all x in $[x_{i-1}, x_i]$. This means F is differentiable for all x in $[x_{i-1}, x_i]$ and, hence, F is continuous on $[x_{i-1}, x_i]$. Therefore, the Mean Value Theorem applies. We have there is a value x_i^* in $[x_{i-1}, x_i]$ with:

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

Rearranging this equation, we get $F(x_i) - F(x_{i-1}) = F'(x_i^*)\,(x_i - x_{i-1}) = f(x_i^*)\Delta x_i$. Hence:

$$\begin{aligned}
F(b) - F(a) &= F(x_n) - F(x_0) \\
&= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \dots + F(x_3) - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0) \\
&= \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right) \\
F(b) - F(a) &= \sum_{i=1}^{n} f(x_i^*)\Delta x_i
\end{aligned}$$

The sum on the right is a Riemann Sum, hence, taking the limit of both sides as $\|\Delta\| \to 0$:

$$F(b) - F(a) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx \quad \checkmark$$

STRATEGY FOR EVALUATING DEFINITE INTEGRALS: To evaluate $\displaystyle\int_a^b f(x)\,dx$:

1. Find an antiderivative of f, F.
2. Evaluate $F(b) - F(a)$.

NOTATION: We write $F(x)|_{x=a}^{x=b} = F(b) - F(a)$ so : $\displaystyle\int_a^b f(x)\,dx = F(x)|_{x=a}^{x=b} = F(b) - F(a)$.

EXAMPLE 1: Evaluate the following using the Fundamental Theorem of Calculus.

1. $\displaystyle\int_{-1}^{3} (2x-1)\,dx$.

$$\text{Ans: } \int_{-1}^{3} (2x-1)\,dx = x^2 - x\Big|_{x=-1}^{x=3} = \big((3)^2 - (3)\big) - \big((-1)^2 - (-1)\big) = 4.$$

2. $\displaystyle\int_{1}^{16} \sqrt{x}\,dx$.

$$\text{Ans: } \int_{1}^{16} \sqrt{x}\,dx = \int_{1}^{16} x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_{x=-1}^{x=16} = \frac{2}{3}(16)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} = 42.$$

3. $\displaystyle\int_{0}^{\pi} \sin(\theta)\,d\theta$.

$$\text{Ans: } \int_{0}^{\pi} \sin(\theta)\,d\theta = -\cos(\theta)|_{\theta=0}^{\theta=\pi} = (-\cos(\pi)) - (-\cos(0)) = 2.$$

4. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc(\theta)\cot(\theta)\,d\theta$.

$$\text{Ans: } \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc(\theta)\cot(\theta)\,d\theta = -\csc(\theta)|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} = -\csc\left(\frac{\pi}{2}\right) + \csc\left(\frac{\pi}{6}\right) = 1.$$

5. $\displaystyle\int_{0}^{\frac{\pi}{8}} \sec^2(2t)\,dt$.

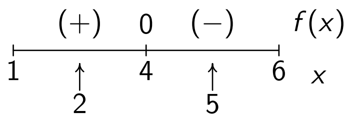
$$\text{Ans: } \int_{0}^{\frac{\pi}{8}} \sec^2(2t)\,dt = \frac{1}{2}\tan(2t)\Big|_{t=0}^{t=\frac{\pi}{8}} = \frac{1}{2}\tan\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan(0) = \frac{1}{2}.$$

EXAMPLE 2: Find and interpret: $\displaystyle\int_1^6 |4x - x^2|\, dx$.

For starters, note that, in general, $\displaystyle\int_a^b |f(x)|\, dx \neq \left|\int_a^b f(x)\, dx\right|$ (do you see why not?)

Recall $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$ hence $|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$.

Hence we make a Sign Diagram for $f(x) = 4x - x^2$ over $[1, 6]$ in order to simplify $|f(x)| = |4x - x^2|$ appropriately:



This means on $[1, 4]$, $f(x) = 4x - x^2 \geq 0$ so $|4x - x^2| = 4x - x^2$. Hence,

$$\int_1^4 \left|4x - x^2\right|\, dx = \int_1^4 \left(4x - x^2\right)\, dx = 2x^2 - \frac{1}{3}x^3\Bigg|_{x=1}^{x=4} = \left(2(4)^2 - \frac{1}{3}(4)^3\right) - \left(2(1)^2 - \frac{1}{3}(1)^3\right) = 9$$

On $[4, 6]$, $f(x) = 4x - x^2 < 0$ so $|4x - x^2| = -(4x - x^2) = -4x + x^2$. Hence,

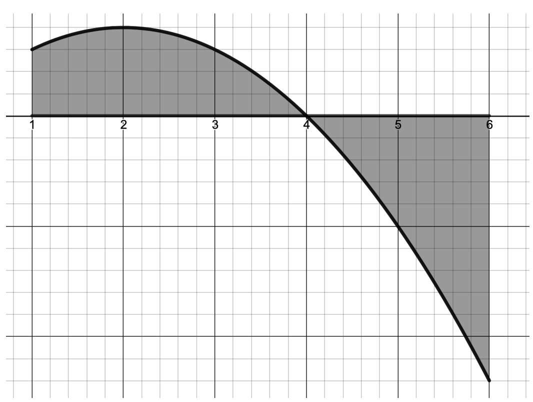
$$\int_4^6 \left|4x - x^2\right|\, dx = \int_4^6 \left(-4x + x^2\right)\, dx = -2x^2 + \frac{1}{3}x^3\Bigg|_{x=1}^{x=4} = \left(-2(6)^2 + \frac{1}{3}(6)^3\right) - \left(-2(4)^2 + \frac{1}{3}(4)^3\right) = \frac{32}{3}$$

Using the additive interval property of definite integrals, we get:

$$\int_1^6 |4x - x^2|\, dx = \int_1^4 |4x - x^2|\, dx + \int_4^6 |4x - x^2|\, dx = 9 + \frac{32}{3} = \frac{59}{3}$$

Geometrically, we have found the **total area** between $y = 4x - x^2$ and the x-axis over $[1, 4]$ is $\dfrac{59}{3}$ units^2.

That is the sum of the (unsigned) areas below.



NOTE: In general:

- $\displaystyle\int_a^b f(x)\, dx$ gives the **net** area between $y = f(x)$ and the x-axis over $[a, b]$.
- $\displaystyle\int_a^b |f(x)|\, dx$ gives the **total** area between $y = f(x)$ and the x-axis over $[a, b]$.

THE FUNDAMENTAL THEOREM OF CALCULUS (ANTIDERIVATIVES AS INTEGRALS):

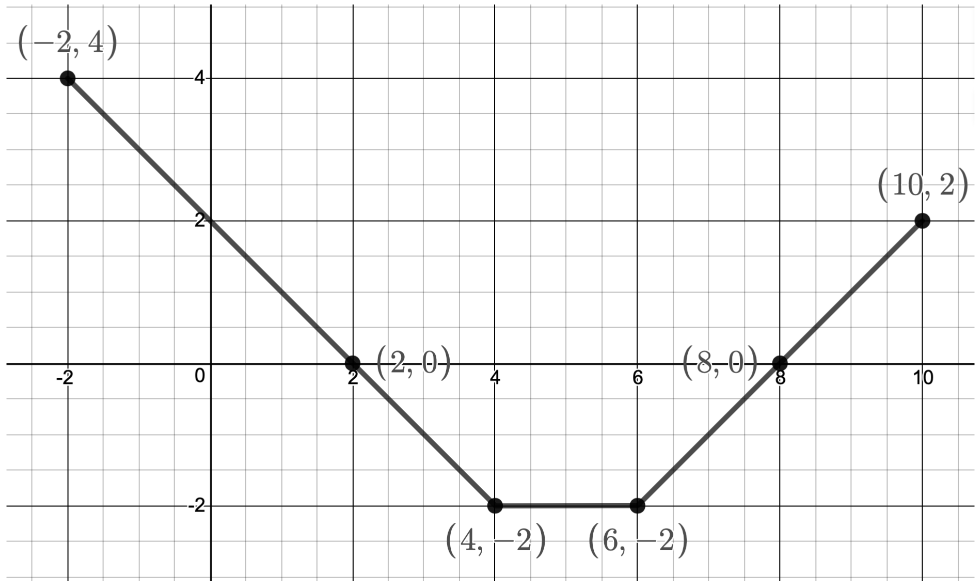
If f is continuous on $[a, b]$ then the function: $F(x) = \int_a^x f(t)\,dt$ is differentiable on (a, b). Moreover:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Before we jump into the proof of this theorem, we first discuss what functions of the form $F(x) = \int_a^x f(t)\,dt$.

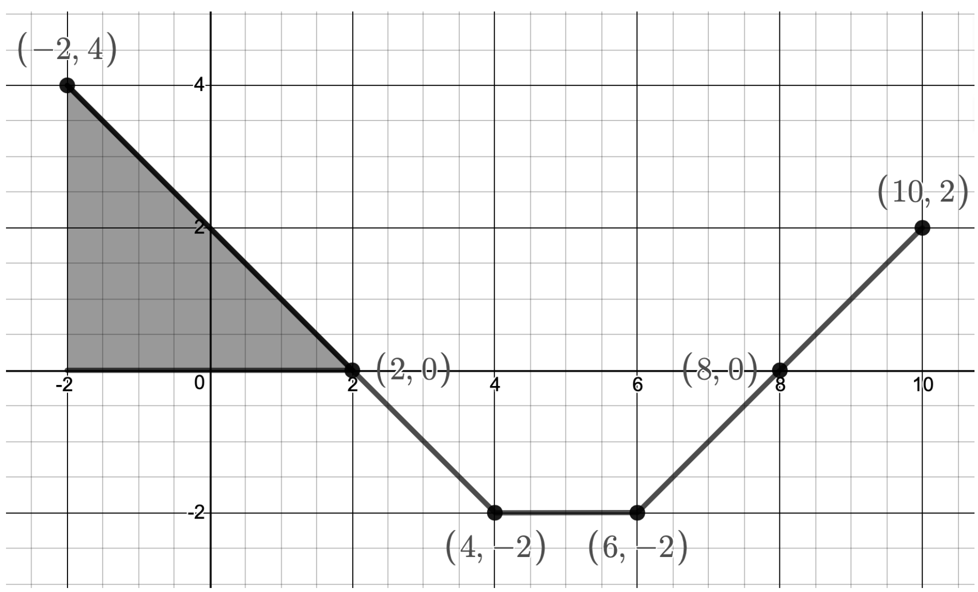
AREA AS YOU GO FUNCTIONS

Consider the graph of $y = f(t)$ below and define $F(x) = \int_{-2}^x f(t)\,dt$ for $-2 \le x \le 10$



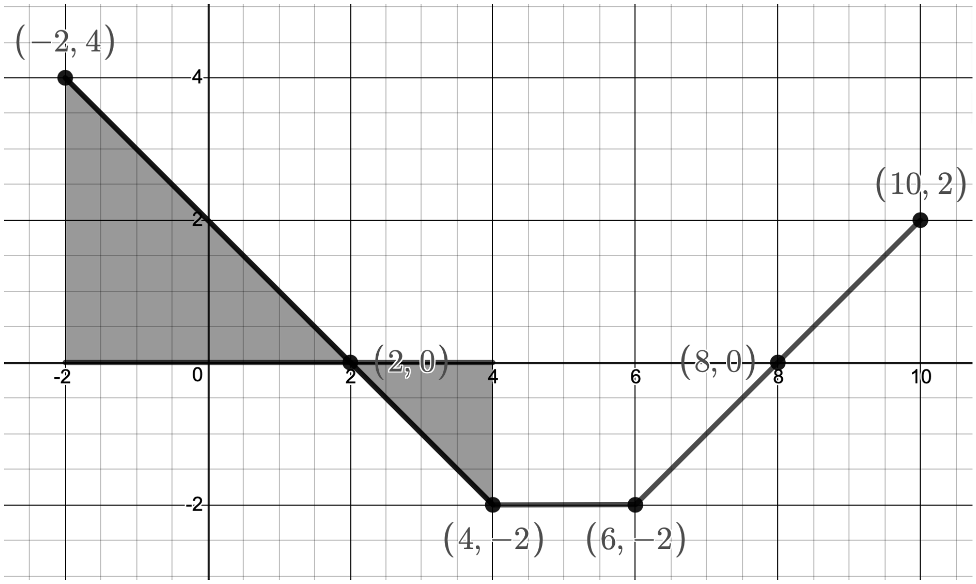
The graph of $y = f(t)$

1. $F(2) = \int_{-2}^2 f(t)\,dt$ which corresponds to the area shaded below so $F(2) = \frac{1}{2}(4)(4) = 8$.



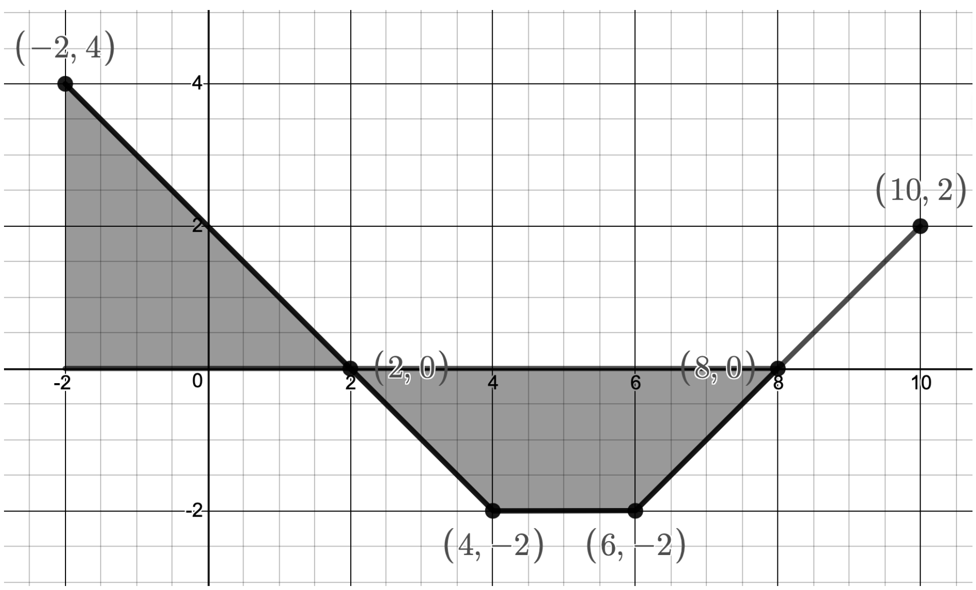
Visualizing $F(2)$

2. $F(4) = \int_{-2}^{4} f(t)\,dt = \frac{1}{2}(4)(4) - \frac{1}{2}(2)(2) = 6$



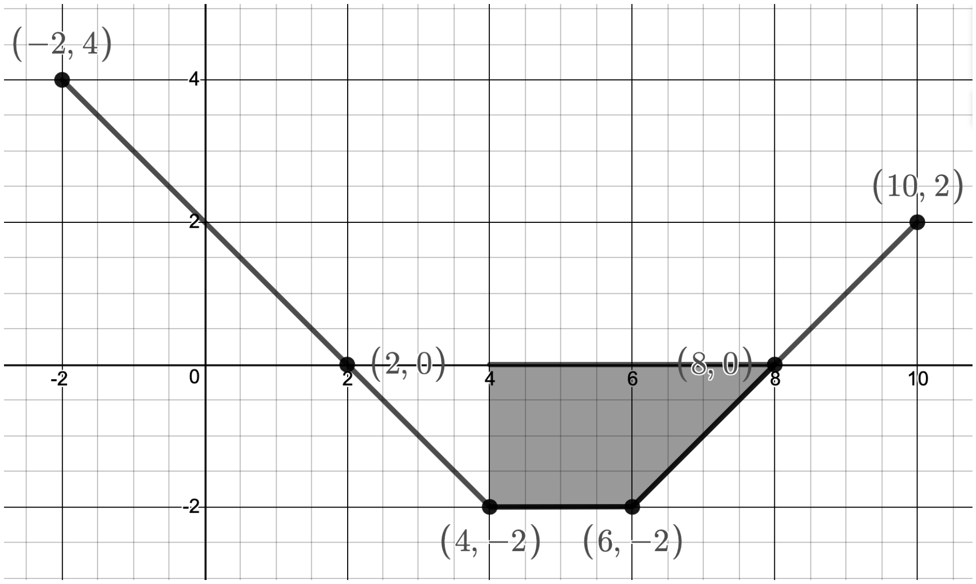
Visualizing $F(4)$

3. $F(8) = \int_{-2}^{8} f(t)\,dt = \frac{1}{2}(4)(4) - \frac{1}{2}(2)(2) - (2)(2) - \frac{1}{2}(2)(2) = 0$



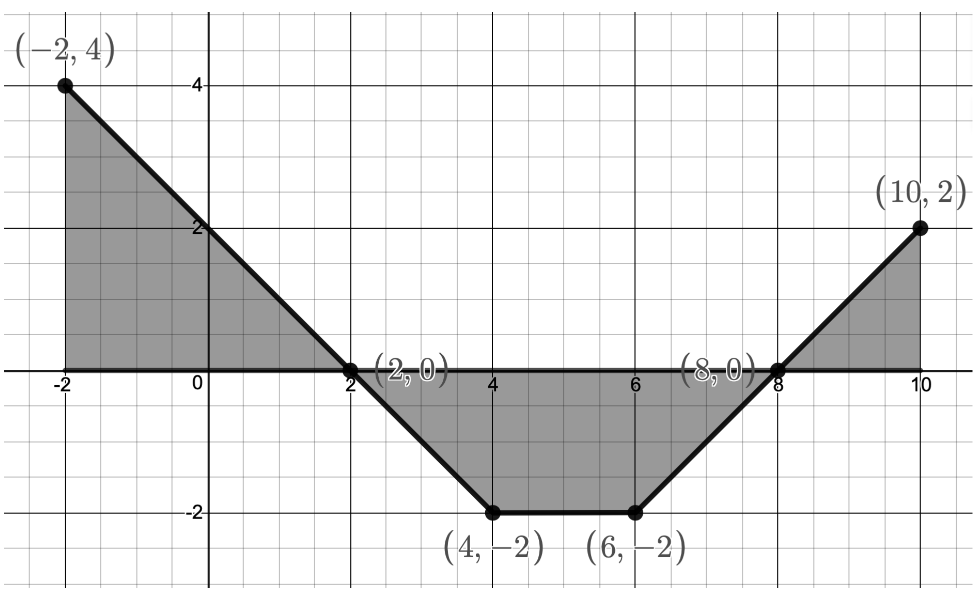
Visualizing $F(8)$

4. $F(8) - F(4) = \int_{-2}^{8} f(t)\,dt - \int_{-2}^{4} f(t)\,dt = 0 - 6 = -6$



Visualizing $F(8) - F(4)$

5. $F(10) = \frac{1}{2}(4)(4) - \frac{1}{2}(2)(2) - (2)(2) - \frac{1}{2}(2)(2) + \frac{1}{2}(2)(2) = 1$



Visualizing $F(10)$

QUESTIONS: Over what interval(s) is F increasing? decreasing?

PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS (ANTIDERIVATIVES AS INTEGRALS):

Suppose f is continuous on $[a, b]$ and let $F(x) = \displaystyle\int_a^x f(t)\,dt$ for x in (a, b). Recall the definition of derivative:

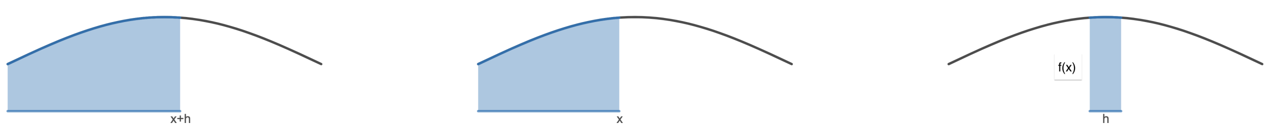
$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\displaystyle\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Focusing on the numerator, we use properties of the definite integral to get:

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt = \int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

Now for a bit of hand-waving[1]... If h is 'small,' then x and $x+h$ are 'close' and as f is continuous $f(t) \approx f(x)$ for all t, $x \le t \le x+h$. Hence, $\displaystyle\int_x^{x+h} f(t)\,dt \approx f(x)\,h$.

Geometrically:



$$\int_a^{x+h} f(t)\,dt \qquad - \qquad \int_a^x f(t)\,dt \qquad = \qquad \int_x^{x+h} f(t)\,dt \approx f(x)\,h$$

Hence,

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\displaystyle\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0}\frac{f(x)\,h}{h} = \lim_{h\to 0} f(x) = f(x)$$

EXAMPLE 3: Use the Fundamental Theorem of Calculus to find the following derivatives:

1. For $F(x) = \displaystyle\int_1^x \sin(t^2)\,dt$, $F'(x) = \dfrac{d}{dx}\displaystyle\int_1^x \sin(t^2)\,dt = \sin(x^2)$.

2. For $G(x) = \displaystyle\int_1^{5x} \sin(t^2)\,dt$, we note $G(x) = F(5x)$ from above.

   Hence, using the chain rule, $G'(x) = F'(5x)D_x[5x] = \sin((5x)^2)(5) = 5\sin\left(25x^2\right)$.

3. For $H(x) = \displaystyle\int_x^{5x} \sin(t^2)\,dt$, we rewrite

   $$H(x) = \int_x^1 \sin(t^2)\,dt + \int_1^{5x} \sin(t^2)\,dt = -\int_1^x \sin(t^2)\,dt + \int_1^{5x} \sin(t^2)\,dt = \int_1^{5x} \sin(t^2)\,dt - \int_1^x \sin(t^2)\,dt$$

   Hence, $H(x) = G(x) - F(x)$ so $H'(x) = G'(x) - F'(x) = 5\sin\left(25x^2\right) - \sin(x^2)$.

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[1] which we'll make more precise in the next section!

The sorts of maneuvers in the previous example generalize to the following theorem:

THEOREM: If f is continuous, then $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

EXAMPLE 4: Find the indicated derivative:

1. $\frac{d}{dx}\int_{2}^{x} \sqrt[4]{1+t^2}\,dt$

$$\text{Ans: } \frac{d}{dx}\int_{2}^{x} \sqrt[4]{1+t^2}\,dt = \sqrt[4]{1+x^2}$$

2. $\frac{d}{dx}\int_{x}^{0} 3t\sin(2t)\,dt$

$$\text{Ans: } \frac{d}{dx}\int_{x}^{0} 3t\sin(2t)\,dt = -\frac{d}{dx}\int_{0}^{x} 3t\sin(2t)\,dt = -3x\sin(2x)$$

3. $\frac{d}{dx}\int_{-\pi}^{x^3} \cos\left(\sqrt[3]{t}\right)\,dt$

$$\text{Ans: } \frac{d}{dx}\int_{-\pi}^{x^3} \cos\left(\sqrt[3]{t}\right)\,dt = \cos\left(\sqrt[3]{x^3}\right) D_x\left[x^3\right] = 3x^2\cos(x)$$

4. $\frac{d}{dx}\int_{-x}^{x} e^{-t^2}\,dt$

$$\text{Ans: } \frac{d}{dx}\int_{-x}^{x} e^{-t^2}\,dt = e^{-x^2}D_x[x] - e^{-(-x)^2}D_x[-x] = e^{-x^2} + e^{-x^2} = 2e^{-x^2}$$

5. $\frac{d}{dx}\int_{0}^{\tan(x)} \frac{1}{t^2+1}\,dt$

$$\text{Ans: } \frac{d}{dx}\int_{0}^{\tan(x)} \frac{1}{t^2+1}\,dt = \frac{1}{\tan^2(x)+1}D_x[\tan(x)] = \frac{\sec^2(x)}{\tan^2(x)+1} = \frac{\sec^2(x)}{\sec^2(x)} = 1$$

HOMEWORK: Section 5.3: 25 - 69 odd, 13, 71 - 103 odd, 107, 117*